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Journal of Algebra 319 (2008) 1913–1931

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

On the ideals of secant varieties to certain rational varieties

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Received 28 September 2006

Available online 7 March 2007

Communicated by Luchezar L. Avramov

Abstract

If $\mathbb{X} \subset \mathbb{P}^n$ is a reduced and irreducible projective variety, it is interesting to find the equations describing the (higher) secant varieties of $\mathbb{X}$. In this paper we find those equations in the following cases:

- $\mathbb{X} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$ is the Segre embedding of the product and n is "large" with respect to the n_i (Theorem 2.4);
- the $\mathbb{X}$ are some "unbalanced" Segre–Veronese embeddings;
- $\mathbb{X}$ is a Del Pezzo surface.

Keywords: Secant varieties; Rational surfaces; Segre varieties; Generators of ideals; Tensor rank

0. Introduction

The study of the higher secant varieties of the Segre varieties has a long and interesting history (see e.g. [ChCi,ChCo,K,Pa,Te,Z]). In addition to its intrinsic beauty and its role in un-

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.01.045

derstanding properties of the projections of algebraic varieties, this study has been influenced by questions from representation theory, coding theory and algebraic complexity theory (see our paper [CGG2] for some recent results as well as a summary of known results, and also [BCS]). Most surprising to us, however, are the connections with the recent work in algebraic statistics (e.g. see [GHKM,GSS]).

Although the major question classically asked about such secant varieties concerned their dimensions, and this is still—by and large—an open and challenging problem, the authors of the paper [GSS] raised some interesting questions about the generators of the defining ideals of such varieties.

Unfortunately, questions about the commutative and homological algebra of the defining ideals of the higher secant varieties of any variety have received only limited attention. Thus, apart from some notable exceptions, there is very little information available about such questions. One family of varieties for which we have rather complete information about the commutative algebra of their higher secant varieties is the family of rational normal curves (i.e. the *Veronese embeddings* of $\mathbb{P}^1$). In this case the ideals in question are generated by the maximal minors of Hankel matrices and one knows not only these generators but also the entire minimal free resolution of these ideals. Similarly, the defining ideals for the higher secant varieties of the *quadratic Veronese embeddings* of $\mathbb{P}^n$ are defined by the (appropriately sized) minors of the generic symmetric matrix of size $(n+1)\times(n+1)$. It follows, thanks to the work of [JPW], that we thus know not only the generators of these ideals but also their minimal free resolutions.

In this paper, however, our main interest is in Segre varieties. In this case it is also well known that if the Segre variety is the embedding of

$$\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\to\mathbb{Y}\subset\mathbb{P}^N,\quad N=(n_1+1)(n_2+1)-1$$

and we let $\sigma_s(\mathbb{Y})$ be the $(s-1)$-secant variety of $\mathbb{Y}$ (i.e. the closure of the union of all the s-secant $\mathbb{P}^{s-1}$'s to $\mathbb{Y}$) then $I_{\sigma_s(\mathbb{Y})}$ is the ideal generated by the $(s+1)\times(s+1)$ minors of the $(n_1+1)\times(n_2+1)$ tensor (i.e. matrix) whose entries are the homogeneous coordinates of $\mathbb{P}^N$, i.e. the ideal of the $(s+1)\times(s+1)$ minors of the generic $(n_1+1)\times(n_2+1)$ matrix. In this case the ideal is rather well understood (see e.g. [L] and also the extensive bibliography given in the book of Weyman [W]).

We will only refer to a small part of this vast subject and recall, e.g. that the ideal $I_{\sigma_s(\mathbb{Y})}$ is a perfect ideal of height

$$\bigl(n_1+1-(s+1)-1\bigr)\cdot\bigl(n_2+1-(s+1)-1\bigr)=(n_1-s-1)(n_2-s-1)$$

in the polynomial ring with $N+1$ variables, with a very well-known resolution.

It follows from this description that all the secant varieties of the Segre embeddings of a product of *two* projective spaces are arithmetically Cohen–Macaulay varieties. Moreover, from the resolution one can also deduce the degree, as well as other significant geometric invariants, of these varieties.

A determinantal formula for the degree was first given by Giambelli. There is, however, a reformulation of this result which we will use (see e.g. [H, p. 244], or [BC, Theorem 6.5], where this lovely reformulation of the Giambelli Formula is attributed to J. Herzog and N.V. Trung):

$$\deg\bigl(\sigma_s(\mathbb{Y})\bigr)=\prod_{i=0}^{n_1-s}\frac{\binom{n_2+1+i}{s}}{\binom{s+i}{s}}.$$

It is worth mentioning that [BC, Theorem 6.9] also have a very nice formula for the Hilbert Series of the coordinate ring of the various secant varieties to $\mathbb{Y}$, but we will not have occasion to use that formula here.

Let us now pass to the case of the Segre embeddings of **more** than two factors. More specifically, let $\mathbb{X} \subset \mathbb{P}^N$ denote the Segre embedding of

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t} \to \mathbb{X} \subset \mathbb{P}^N, \quad N = \prod_{i=1}^{t}(n_i + 1) - 1,\ t \geqslant 3,$$

where we usually assume that $n_1 \leqslant \cdots \leqslant n_t$.

If we let T be the $(n_1 + 1) \times \cdots \times (n_t + 1)$ tensor whose entries are the homogeneous coordinates in $\mathbb{P}^N$, then it is well known that the ideal of $\mathbb{X}$ is given by the 2×2 minors of T. It is natural to ask if there is some way to use the tensor T to get information about the higher secant varieties of $\mathbb{X}$.

If we partition $\{1, \dots, t\}$ into two subsets (say $\{1, \dots, \ell\}$ and $\{\ell + 1, \dots, t\}$, to keep the notation simple) then we can form the composition

$$\left(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}\right) \times \left(\mathbb{P}^{n_{\ell+1}} \times \cdots \times \mathbb{P}^{t}\right) \to \mathbb{P}^a \times \mathbb{P}^b$$

where $a = \prod_{i=1}^{\ell}(n_i + 1) - 1$, $b = \prod_{i=\ell+1}^{t}(n_i + 1) - 1$, followed by

$$\phi : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^N, \quad N \text{ as above}.$$

Clearly $\phi(\mathbb{P}^a \times \mathbb{P}^b) \supseteq \mathbb{X}$ and hence

$$\sigma_s(\mathbb{X}) \subseteq \sigma_s\left(\phi\left(\mathbb{P}^a \times \mathbb{P}^b\right)\right).$$

Thus, the $(s + 1) \times (s + 1)$ minors of the matrix associated to the embedding ϕ will all vanish on $\sigma_s(X)$. That matrix, written in terms of the coordinates of the various $\mathbb{P}^{n_i}$ is called a *flattening* of the tensor T.

We can perform a *flattening* of T for every partition of $\{1, \dots, t\}$ into two subsets. The $(s + 1) \times (s + 1)$ minors of all of these flattenings will give us equations which vanish on $\sigma_s(\mathbb{X})$. In [GSS] it was conjectured that, at least for $s = 2$, these equations are precisely the generators for the ideal $I_{\sigma_2(\mathbb{X})}$ of $\sigma_2(\mathbb{X})$. The conjecture was proved in [LM] for the special case of $t = 3$ (and set theoretically for all t's). More recently, Allman and Rhodes [AR] proved the conjecture for up to five factors. Landsberg and Weyman [LW] have found the generators for the defining ideals of secant varieties for the Segre varieties in the following cases: all secant varieties for $\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^n$ for all m, n; the secant plane varieties for any Segre variety with three factors. The proofs by Landsberg et al. use representation theoretic methods. To our knowledge, these are the only known results describing the ideals of higher secant variety for infinite families of Segre embeddings with more than 2 factors.

Note that for $s > 2$ one cannot expect, in general, that the ideals $I_{\sigma_s(\mathbb{X})}$ are generated by the $(s + 1) \times (s + 1)$ minors of flattenings of T. Indeed, in many cases there are no such minors, as the following example illustrates.

Example 0.1. Let $\mathbb{X} = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (5-times). The Segre embedding gives us a $2 \times 2 \times 2 \times 2 \times 2$ box and the various flattenings will give us

(i) 10 (4×8) matrices; and
(ii) 5 (2×16) matrices.

The largest minors we can look at for these flattenings are the 4×4 minors of the first set of matrices and those will give us (some) equations for $\sigma_3(\mathbb{X})$ and for no higher secant variety of $\mathbb{X}$. But, $\mathbb{X}$ (of dimension 5) lives in $\mathbb{P}^{31}$ and one sees, by a simple dimension count, that $\sigma_4(\mathbb{X})$ and $\sigma_5(\mathbb{X})$ definitely lie on some hypersurfaces of $\mathbb{P}^{31}$.

Nevertheless, in the second section of this paper we will show that infinitely often the ideal of $\sigma_s(\mathbb{X})$ can be described by the $(s+1) \times (s+1)$ minors of **one** flattening of T (see Theorem 2.4). It follows immediately that these $\sigma_s(\mathbb{X})$ are arithmetically Cohen–Macaulay schemes with a well-known minimal free resolution for their defining ideals. As a consequence we obtain a method for finding the degrees of these secant varieties as well as other numerical invariants that can be calculated from the minimal free resolution (e.g. the Hilbert polynomial).

In the third section we study some Segre–Veronese varieties. These are (special) linear sections of Segre varieties. We apply the results of Section 2 to these varieties and are able to find the ideals of their secant varieties in many cases.

In the final section of the paper we consider Del Pezzo varieties. We give a complete description of the ideals of all of their secant varieties.

1. Preliminaries

We will always work over an algebraically closed field K of characteristic 0.

We recall the notion of *higher secant variety*.

Definition 1.1. Let $\mathbb{X} \subseteq \mathbb{P}^N$ be a closed irreducible projective variety of dimension n. The sth *higher secant variety* of $\mathbb{X}$, denoted $\sigma_s(\mathbb{X})$, is the subvariety of $\mathbb{P}^N$ which is the closure of the union of all linear spaces spanned by s linearly independent points of $\mathbb{X}$.

For $\mathbb{X}$ as above, a simple parameter count gives the following inequality involving the dimension of $\sigma_s(\mathbb{X})$:

$$\dim \sigma_s(\mathbb{X}) \leqslant \min\{N, sn + s - 1\}. \tag{1}$$

Naturally, one "expects" the inequality should, in general, be an equality.

When $\sigma_s(\mathbb{X})$ does not have the "expected" dimension, $\mathbb{X}$ is said to be $(s-1)$-*defective*, and the positive integer

$$\delta_{s-1}(\mathbb{X}) = \min\{N, sn + s - 1\} - \dim \sigma_s(\mathbb{X})$$

is called the $(s-1)$-*defect* of $\mathbb{X}$.

We will have occasion to consider a generalization of the higher secant varieties of a variety. These are the *Grassmann secant varieties*, whose definition we now recall.

Definition 1.2. Let $\mathbb{X} \subseteq \mathbb{P}^N$ be a reduced and irreducible projective variety of dimension n, s any integer $\leqslant N$.

For k any integer, $0 \leqslant k \leqslant s-1$, the $(k, s-1)$-*Grassmann secant variety of* $\mathbb{X}$ (denoted $\mathrm{Sec}_{k,s-1}(\mathbb{X})$) is the Zariski closure, in the Grassmannian of k-dimensional linear subspaces of $\mathbb{P}^N$ (which we will denote $\mathbb{G}(k, N)$) of the set

$$\bigl\{l \in \mathbb{G}(k, N) \bigm| l \text{ is a subspace of the span of } s \text{ independent points of } \mathbb{X}\bigr\}.$$

In case $k = 0$ we get $\mathrm{Sec}_{0,s-1}(\mathbb{X}) = \sigma_s(\mathbb{X})$.

As a generalization of the analogous result for the higher secant varieties, one always has the inequality

$$\dim \mathrm{Sec}_{k,s-1}(\mathbb{X}) \leqslant \min\bigl\{sn + (k+1)(s-k-1), (k+1)(N-k)\bigr\},$$

with equality being what is generally "expected."

When $\mathrm{Sec}_{k,s-1}(\mathbb{X})$ does not have the expected dimension then we say that $\mathbb{X}$ is $(k, s-1)$-*defective* and in this case we define the $(k, s-1)$-defect of $\mathbb{X}$ as the number:

$$\delta_{k,s-1}(\mathbb{X}) = \min\bigl\{sn + (k+1)(s-k-1), (k+1)(N-k)\bigr\} - \dim \mathrm{Sec}_{k,s-1}(\mathbb{X}).$$

(For general information on these defectivities see [ChCo] and [DF].)

In his paper [Te2], Terracini gives a link between these two kinds of defectivity for a variety $\mathbb{X}$ as above (see [DF] for a modern proof):

Proposition 1.3 *(Terracini). Let $\mathbb{X} \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n. Let $\psi : \mathbb{X} \times \mathbb{P}^k \to \mathbb{P}^{(k+1)(N+1)-1}$ be the* (*usual*) *Segre embedding.*

Then $\mathbb{X}$ is $(k, s-1)$-defective with defect $\delta_{k,s-1}(\mathbb{X}) = \delta$ if and only if $\psi(\mathbb{X} \times \mathbb{P}^k)$ is $(s-1)$-defective with $(s-1)$-defect $\delta_{s-1}(\mathbb{X} \times \mathbb{P}^k) = \delta$.

Finally we wish to give a simple, but useful, lemma which we have been unable to find in the literature.

Lemma 1.4. *Let $\mathbb{X} \subset \mathbb{Y} \subset \mathbb{P}^N$ be reduced irreducible projective varieties. Suppose that for some integer s we have*:

$$\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}).$$

Then

$$\sigma_{s+1}(\mathbb{X}) = \sigma_{s+1}(\mathbb{Y}).$$

Proof. One inclusion is clear, so suppose $P \in \sigma_{s+1}(\mathbb{Y})$. Then we can find $s+1$ linearly independent points of $\mathbb{Y}$, call them $Q_0, Q_1, \ldots, Q_s$, such that

$$P = \alpha_0 Q_0 + \alpha_1 Q_1 + \cdots + \alpha_s Q_s.$$

Clearly $P' = \alpha_1 Q_1 + \cdots + \alpha_s Q_s \in \sigma_s(\mathbb{Y}) = \sigma_s(\mathbb{X})$, so we can write

$$P' = \beta_1 R_1 + \cdots + \beta_s R_s$$

where $R_1, \dots, R_s$ are linearly independent points in $\mathbb{X}$. Thus, we can rewrite P as

$$P = \alpha_0 Q_0 + \beta_1 R_1 + \cdots + \beta_s R_s.$$

Now consider

$$P'' = \alpha_0 Q_0 + \beta_1 R_1 + \cdots + \beta_{s-1} R_{s-1}.$$

With the same reasoning as above, we can write

$$P'' = \gamma_0 T_0 + \cdots + \gamma_{s-1} T_{s-1}$$

where $T_0, \dots, T_{s-1}$ are linearly independent points of $\mathbb{X}$.

Putting this all together we get

$$P = \gamma_0 T_0 + \cdots + \gamma_{s-1} T_{s-1} + \beta_s R_s$$

and the points $T_0, \dots, T_{s-1}, R_s$ are all points in $\mathbb{X}$. That finishes the proof. $\square$

2. The main idea: The unbalanced case

As we mentioned earlier, we will be interested in finding Segre varieties $\mathbb{X}$ for which some higher secant variety is described by the appropriate sized minors of **one** flattening of the tensor whose 2×2 minors describe $\mathbb{X}$. We will consider Segre embeddings of products

$$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^{n},$$

where $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_t \leqslant n$ (often $n \gg n_t$ hence the term "*unbalanced*").

The following easy example (see [P], and for the case of the secant line variety see also [LM]) will illustrate the main idea in what follows.

Example 2.1. Consider the Segre varieties $\mathbb{X}$ given by embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{4n+3}$, $n \geqslant 2$. The ideal of $\mathbb{X}$ is given by the 2×2 minors of a $2 \times 2 \times (n+1)$ tensor T of indeterminates (the coordinates of $\mathbb{P}^{4n+3}$).

$$\begin{array}{cccccccccc}
 & v_{000} & --- & v_{001} & --- & v_{002} & & \cdots & & v_{00n} \\
 & | & \searrow & & \searrow & & & & \searrow & \\
T = & v_{100} & & v_{010} & --- & v_{011} & --- & \cdots & --- & v_{01n} \\
 & & \searrow & | & & | & & & & | \\
 & & & v_{110} & --- & v_{111} & --- & \cdots & --- & v_{11n}
\end{array}$$

Consider the $4 \times (n+1)$ matrix M obtained by flattening this tensor, i.e. by using the composition:

$$\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \times \mathbb{P}^n \to \mathbb{P}^3 \times \mathbb{P}^n \to \mathbb{P}^{4n+3},$$

$$M = \begin{pmatrix} v_{000} & v_{001} & v_{002} & --- & v_{00n} \\ v_{100} & v_{101} & v_{102} & --- & v_{10n} \\ v_{110} & v_{111} & v_{112} & --- & v_{11n} \\ v_{010} & v_{011} & v_{012} & --- & v_{01n} \end{pmatrix}.$$

The ideal generated by the 2×2 minors of M is the ideal of the Segre variety $\mathbb{Y}$ given by embedding $\mathbb{P}^3 \times \mathbb{P}^n$ into $\mathbb{P}^{4n+3}$. Trivially $\mathbb{Y}$ contains $\mathbb{X}$.

Now consider the ideal generated by the 3×3 minors of M. This is well known to be the ideal of $\sigma_2(\mathbb{Y})$. Of course those minors also vanish on $\sigma_2(\mathbb{X})$. Since the matrix M has generic entries, we know that its 3×3 minors generate a prime ideal of height $(4-3+1)(n+1-3+1) = 2n-2 = \operatorname{codim} \sigma_2(\mathbb{Y})$ which, in fact, is defective (its expected codimension is $2n-4$). Thus the dimension of $\sigma_2(\mathbb{Y})$ is $2n+5$.

But we know that $\dim \sigma_2(\mathbb{X}) = 2n+5$ (Segre varieties with three or more factors always have σ_2 of the expected dimension, see [CGG2]), hence $\sigma_2(\mathbb{X}) = \sigma_2(\mathbb{Y})$. By the lemma above, this implies that (for $t \geqslant 3$) also $\sigma_t(\mathbb{X}) = \sigma_t(\mathbb{Y})$. In this example, the only other relevant t is $t=3$ (for $t=4$ we have $\sigma_4(\mathbb{X}) = \mathbb{P}^{4n+3}$) and thus $I_{\sigma_3(\mathbb{X})}$ is generated by the 4×4 minors of M (only relevant when $n \geqslant 3$). This ideal is an ideal of height $(n+1-4+1) = n-2$ and so $\sigma_3(\mathbb{X})$ is defective, having dimension $3n+5$ instead of $3n+7$. Since $\sigma_3(\mathbb{X})$ is defined by the maximal minors of a $4 \times (n+1)$ matrix, we can also say that its degree is $\binom{n+1}{3}$ and it is arithmetically Cohen–Macaulay.

The point of this example is, we hope, clear: it sometimes happens that a tth secant variety for the Segre product of three or more projective spaces is the same as the tth secant variety of a Segre product with only two factors. Inasmuch as we have abundant information about Segre products with two factors, this gives us a way to get information about Segre products with more than two factors.

Our first task is to find more times when the behavior in Example 2.1 occurs. This is the content of the following lemma.

Lemma 2.2. *Let $V \subset \mathbb{P}^N$ be a variety such that $\mathrm{Sec}_{s-1,s-1}(V) = \mathbb{G}(s-1,N)$. Consider the Segre embedding $\mathbb{Y}$ of $\mathbb{P}^N \times \mathbb{P}^n$ into $\mathbb{P}^M$, $M = Nn + N + n$.*

If $\mathbb{X}$ is the image of $V \times \mathbb{P}^n$ into $\mathbb{P}^M$, then $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y})$.

Proof. Let $\phi : \mathbb{P}^N \times \mathbb{P}^n \to \mathbb{P}^M$ be the Segre embedding. Consider a general secant $\mathbb{P}^{s-1}$ to $\mathbb{Y}$ (the image of ϕ) and call it H. Then,

$$\mathbb{P}^{s-1} \cong H = \left\langle \phi(A_0, B_0), \ldots, \phi(A_{s-1}, B_{s-1}) \right\rangle,$$

with $A_i \in \mathbb{P}^N$, $B_i \in \mathbb{P}^n$, generic points in their spaces. For all $\lambda_0, \ldots, \lambda_{s-1} \in K$ we want to check that the point:

$$P_{\underline{\lambda}} = \lambda_0 \phi(A_0, B_0) + \cdots + \lambda_{s-1} \phi(A_{s-1}, B_{s-1})$$

is in $\sigma_s(\mathbb{X})$. We will be done if we find points $C_0, \ldots, C_{s-1}$ in V and $D_0, \ldots, D_{s-1}$ in $\mathbb{P}^n$ such that:

$$P_{\underline{\lambda}} = \phi(C_0, D_0) + \cdots + \phi(C_{s-1}, D_{s-1}).$$

Since $\mathrm{Sec}_{s-1,s-1}(V) = \mathbb{G}(s-1, V)$ and the points A_i are generic in $\mathbb{P}^N$ we can choose the C_i's in V such that

$$\langle C_0, \dots, C_{s-1}\rangle = \langle A_0, \dots, A_{s-1}\rangle,$$

and so we can write

$$A_i = \sum_{j=0}^{s-1} a_j^{(i)} C_j.$$

Since ϕ is a bilinear map, we obtain:

$$\begin{aligned}
P_{\underline{\lambda}} &= \lambda_0\phi(A_0, B_0) + \cdots + \lambda_{s-1}\phi(A_{s-1}, B_{s-1}) \\
&= \lambda_0\phi\left(\sum_{j=0}^{s-1} a_j^{(0)} C_j, B_0\right) + \cdots + \lambda_{s-1}\phi\left(\sum_{j=0}^{s-1} a_j^{(s-1)} C_j, B_{s-1}\right) \\
&= \lambda_0\left[\sum_{j=0}^{s-1} a_j^{(0)} \phi(C_j, B_0)\right] + \cdots + \lambda_{s-1}\left[\sum_{j=0}^{s-1} a_j^{(s-1)} \phi(C_j, B_{s-1})\right] \\
&= \sum_{j=0}^{s-1} \phi\big(C_j, \lambda_0 a_j^{(0)} B_0\big) + \cdots + \sum_{j=0}^{s-1} \phi\big(C_j, \lambda_{s-1} a_j^{(s-1)} B_{s-1}\big) \\
&= \sum_{j=0}^{s-1} \phi\big(C_j, \lambda_0 a_j^{(0)} B_0 + \cdots + \lambda_{s-1} a_j^{(s-1)} B_{s-1}\big) \\
&= \phi(C_0, D_0) + \cdots + \phi(C_{s-1}, D_{s-1})
\end{aligned}$$

where $D_j := \lambda_0 a_j^{(0)} B_0 + \cdots + \lambda_{s-1} a_j^{(s-1)} B_{s-1} \in \mathbb{P}^n$ and the C_i's are in V, and we are done. $\square$

We now look for times when the hypothesis of Lemma 2.2 are satisfied. To that end, consider Segre varieties $\mathbb{X} \subset \mathbb{P}^M$, $M = [\prod_{i=1}^t (n_i+1)](n+1) - 1$, given by embedding $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$ into $\mathbb{P}^M$, and also the Segre variety $\mathbb{X}'$ given by embedding $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$ into $\mathbb{P}^N$, $N = \prod_{i=1}^t (n_i+1) - 1$. We can consider $\mathbb{X}$ as obtained by composing the map, which is the Segre embedding on the first factor and the identity on the second factor, with the Segre embedding $\mathbb{Y}$ of $\mathbb{P}^N \times \mathbb{P}^n$ into $\mathbb{P}^M$. I.e. we have:

$$\begin{tikzcd}
\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n \arrow[rr] \arrow[d] & & \mathbb{P}^M \\
\mathbb{X}' \times \mathbb{P}^n & \subset & \mathbb{P}^N \times \mathbb{P}^n \arrow[u]
\end{tikzcd}$$

We will apply the following lemma to $\mathbb{X}'$.

Lemma 2.3. *Let $V \subset \mathbb{P}^N$ be a reduced and non-degenerate variety. Then V has the property that* $\mathrm{Sec}_{s-1,s-1}(V) = \mathbb{G}(s-1, N)$ *if and only if*:

$$\operatorname{codim}(V) + 1 \leqslant s.$$

Proof. If $s \leqslant \operatorname{codim}(V)$, then a generic $\mathbb{P}^{s-1} \subset \mathbb{P}^N$ will not intersect V, hence in this case $\mathrm{Sec}_{s-1,s-1}(V) \neq \mathbb{G}(s-1, N)$.

Now let $\operatorname{codim}(V) + 1 = s$; since V is reduced and non-degenerate, a general linear subspace of $\mathbb{P}^N$ of dimension $\operatorname{codim}(V)$ will meet V in $\deg V$ distinct points. Again, since V is non-degenerate, $\deg V \geqslant \operatorname{codim}(V) + 1$. Thus, since $s - 1 = \operatorname{codim}(V)$, a generic $\mathbb{P}^{s-1}$ of $\mathbb{P}^N$ meets V in at least s points. Hence, such a $\mathbb{P}^{s-1}$ is definitely an s-secant linear space to V. It follows that for this s we have

$$\mathrm{Sec}_{s-1,s-1}(V) = \mathbb{G}(s-1, N). \tag{$*$}$$

If now we choose s so that $s - 1 > \operatorname{codim}(V)$ then a generic $\mathbb{P}^{s-1}$ of $\mathbb{P}^N$ will meet V in a variety of dimension > 0 and hence will certainly be a secant $\mathbb{P}^{s-1}$ to V. Thus, $(*)$ is also true for such an s and Lemma 2.3 has been verified. □

Notice that for $V = \mathbb{X}'$ we get $\mathrm{Sec}_{s-1,s-1}(\mathbb{X}') = \mathbb{G}(s-1, N) \Leftrightarrow s \geqslant N - \sum_{i=1}^{t} n_i + 1$.

With all the preliminary observations being established, we are now ready to prove the main result of this section.

Theorem 2.4. *Let $\mathbb{X}$ be the Segre embedding*

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^{n} \to \mathbb{X} \subset \mathbb{P}^M, \quad M = (n+1)\left(\prod_{i=1}^{t}(n_i+1)\right) - 1,$$

and let $\mathbb{Y}$ be the Segre embedding of $\mathbb{P}^N \times \mathbb{P}^n$ in $\mathbb{P}^M$, $N = \prod_{i=1}^{t}(n_i+1) - 1$. Let $n \geqslant N - \sum_{i=1}^{t} n_i + 1$.

Then:

(1) *for $2 \leqslant s \leqslant N - \sum_{i=1}^{t} n_i$, $\sigma_s(\mathbb{X}) \neq \sigma_s(\mathbb{Y})$ and $\sigma_s(\mathbb{X})$ has the expected dimension*;
(2) *for $s = N - \sum_{i=1}^{t} n_i + 1$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) \neq \mathbb{P}^M$ and $\sigma_s(\mathbb{X})$ has the expected dimension*;
(3) *for $N - \sum_{i=1}^{t} n_i + 1 < s \leqslant \min\{n, N\}$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) \neq \mathbb{P}^M$ and $\sigma_s(\mathbb{X})$ is defective with $\delta_{s-1}(\mathbb{X}) = s^2 - s(N - \sum_{i=1}^{t} n_i + 1)$*;
(4) *for $s \geqslant \min\{n, N\} + 1$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) = \mathbb{P}^M$*;
(5) *in cases* (2) *and* (3) *above, the ideal of $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y})$ is generated by the $(s+1) \times (s+1)$ minors of an $(n+1) \times (N+1)$ matrix of indeterminates.*

It follows that, in cases (2) *and* (3), *$\sigma_s(\mathbb{X})$ is a.C.M. and a minimal free resolution of its defining ideal is given by the Eagon–Northcott complex.*

Proof. (2) First notice that from Lemmas 2.2 and 2.3, the equality $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y})$ is immediate. We have already mentioned that one knows the dimension of $\sigma_s(\mathbb{Y})$ for any s and a simple calculation reveals that the dimension we obtain for $\sigma_s(\mathbb{X})$ is that which is expected.

As for (1), the hypothesis $n \geqslant N - \sum_{i=1}^{t} n_i + 1$ guarantees that $\sigma_s(\mathbb{Y}) \neq \mathbb{P}^M$. The result then follows immediately from (2) and our knowledge of the dimensions of $\sigma_s(\mathbb{Y})$.

The equality of $\sigma_s(\mathbb{X})$ and $\sigma_s(\mathbb{Y})$ in (3) and (4) is again guaranteed by Lemmas 2.2 and 2.3. Once again we use the fact that the dimensions of the $\sigma_s(\mathbb{Y})$ are known and a simple calculation gives: the defectivity in the range described in (3); the equality in the range described in (4).

(5) is, again, an immediate application of our characterization of the ideal of $\sigma_s(\mathbb{Y})$. The closing statement of the theorem also follows from this characterization. $\square$

Remark 2.5. If we continue with the notation of Theorem 2.4, and suppose that $n = N - \sum_{i=i}^{t} n_i$ and $s = n + 1$ then $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) = \mathbb{P}^M$ (exactly as in part (4) of Theorem 2.4). Moreover, one has the additional fact that

$$\dim \sigma_s(\mathbb{X}) = s \dim(\mathbb{X}) + (s - 1)$$

and hence that $\dim \sigma_t(\mathbb{X})$ is the expected dimension for *every* t.

As a consequence of Theorem 2.4 we have the following:

Corollary 2.6. *Let $\mathbb{X} \subset \mathbb{P}^M$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^n$, $m \leqslant n$ (hence $M = 2nm + 2n + 2m + 1$).*

(i) *If $n = m$, then $\sigma_s(\mathbb{X})$ has the expected dimension for all s;*
(ii) *if $n = m + 1$, then $\sigma_s(\mathbb{X})$ has the expected dimension for all s;*
(iii) *if $n > m + 1$, and*
- *$2 \leqslant s \leqslant m + 1$, then $\sigma_s(\mathbb{X}) \neq \mathbb{P}^M$ has the expected dimension;*
- *$m + 2 \leqslant s \leqslant \min\{2m + 1, n\}$, then $\sigma_s(\mathbb{X})$ is defective with $\delta_{s-1}(\mathbb{X}) = s^2 - s(m + 2)$;*
- *$s > \min\{2m + 1, n\}$, then $\sigma_s(\mathbb{X}) = \mathbb{P}^M$.*

Proof. (i) is immediate from Remark 2.5. (ii) and (iii) all follow from the various parts of Theorem 2.4. $\square$

Notice that partial results in this case can be found in [J], see [CGG2, p. 282]. It is interesting to compare our results with those found by [CS,LM,LW].

Example 2.7. The family of Segre varieties $\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^n$, has been also considered in [LW]. These authors show that the ideal of $\sigma_s(\mathbb{X})$ is generated by the $(s + 1) \times (s + 1)$ minors of the flattenings of the tensor T giving the embedding of $\mathbb{X}$. [LW] do not discuss the dimensions of the secant varieties to members of this family and, consequently, do not mention their defectivities. Note, however, that there are really only two flattenings to consider for members of this family (the third one has no 3×3 minors). In fact, for $m + 1 \leqslant s \leqslant \min\{n, 2m + 1\}$, $I_{\sigma_s(\mathbb{X})}$ is the ideal of the $(s + 1) \times (s + 1)$ minors of a single flattening of T. The proofs in [LW] rely on a subtle analysis using representation theory.

In any case, when we have that $\sigma_s(\mathbb{X})$ is determinantal, then it is given by the $(s + 1) \times (s + 1)$ minors of a single flattening. We can then apply the Giambelli formula in order to get the degree of $\sigma_s(\mathbb{X})$. For example, if we consider the case $n = m + 1$ and let $\mathbb{X}_m$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^{m+1}$, then $\sigma_{m+1}(\mathbb{X})$ has ideal generated by the $(m+2)\times(m+2)$ minors of a $2(m+1)\times(m+2)$ matrix and hence

$$\deg\bigl(\sigma_{m+1}(\mathbb{X}_m)\bigr) = \binom{2m+2}{m+1}.$$

Cox and Sidman (in [CS, Theorem 5.1]) give a formula for the degree of $\sigma_2(\mathbb{X}_m)$. For s such that $3 \leqslant s \leqslant m$ we are not aware of any method to calculate the degree of $\sigma_s(\mathbb{X}_m)$.

Now let us consider the case of four factors $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} \times \mathbb{P}^{n_4} \to \mathbb{X} \subset \mathbb{P}^M$, $M = [\prod_{i=1}^4 (n_i+1)] - 1$ (and, as always, $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4$).

In this case [LW] prove that the ideal of $\sigma_2(\mathbb{X})$ is generated by the 3×3 minors of all the flattenings of the tensor describing $\mathbb{X}$.

If we consider the function

$$N(n_1,n_2,n_3) = (n_1+1)(n_2+1)(n_3+1) - (n_1+n_2+n_3)$$

then our results apply to all those $\mathbb{X}$ (as above) for which $n_4 \geqslant N(n_1,n_2,n_3)$. In this case we have:

(1) a complete description of the dimensions of $\sigma_s(\mathbb{X})$ for every s;
(2) if, in addition, $s \geqslant N(n_1,n_2,n_3)$ then the ideal of $\sigma_s(\mathbb{X})$ is generated by the minors of one flattening of the tensor describing $\mathbb{X}$ and so we also know the finite free resolution of this ideal.

These results apply, for example, to:

$$\begin{aligned}
&\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \quad \text{for } n \geqslant 5;\\
&\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^n \quad \text{for } n \geqslant 8;\\
&\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^n \quad \text{for } n \geqslant 13;\\
&\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^n \quad \text{for } n \geqslant 21.
\end{aligned}$$

It is also possible to apply Theorem 2.4 in order to obtain results on Grassmann defectivity.

Corollary 2.8. *Let $\mathbb{X}, n, N$ be as in Theorem* 2.4*, and let $\mathbb{X}_i$ be the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \widehat{\mathbb{P}^{n_i}} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^n$, $i = 1,\dots,t$. Then for $N - \sum_{i=1}^t n_i + 1 \leqslant s \leqslant \min\{n, N\}$, we have that* $\mathrm{Sec}_{n_i,s-1}(\mathbb{X}_i)$ *is defective, with $\delta_{n_i,s-1} = s^2 - s(N - \sum_{i=1}^t n_i + 1)$, while* $\mathrm{Sec}_{n_i,s-1}(\mathbb{X}_i)$ *has the expected dimension for all other values of s.*

Proof. The corollary is a direct consequence of Theorem 2.4 and Proposition 1.3. □

Corollary 2.9. *Let $\mathbb{X}_n$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Then* $\mathrm{Sec}_{m,s-1}(\mathbb{X}_n)$ *is defective if and only if $n > m+1$ and $m+2 \leqslant s \leqslant \min\{2m+1, n\}$. Moreover, in this case, $\delta_{m,s-1}(\mathbb{X}_n) = s^2 - s(m+2)$.*

Proof. This follows directly from Corollaries 2.6 and 2.8. □

In the following example we will consider how to use the "unbalanced case" idea also when we are not able to describe the ideal of the secant variety completely.

Example 2.10. In this example we would like to consider the following family of Segre varieties, this time with four factors:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{X}(n) \subset \mathbb{P}^N, \quad N = 4n^2 + 8n + 3.$$

Theorem 2.4 does not apply to members of this family. Thus, we cannot say that any secant variety for this family has equations derived from **one** flattening of the tensor describing $\mathbb{X}(n)$. Nevertheless, it is possible to show, using flattenings, that $\sigma_{2n+1}(\mathbb{X}(n))$ is defective.

A quick check shows that the "expected dimension" of $\sigma_{2n+1}(\mathbb{X}(n))$ is $4n^2 + 8n + 2$, i.e. we expect that $\sigma_{2n+1}(\mathbb{X}(n))$ is a hypersurface of $\mathbb{P}^N$. But, if we group the factors of $\mathbb{X}(n)$ above as

$$\left(\mathbb{P}^1 \times \mathbb{P}^n\right) \times \left(\mathbb{P}^1 \times \mathbb{P}^n\right)$$

and then permute the $\mathbb{P}^1$'s, we obtain two distinct embeddings of $\mathbb{P}^{2n+1} \times \mathbb{P}^{2n+1} \to \mathbb{P}^N$ and the determinants of the resulting matrices of size $(2n+2) \times (2n+2)$ give us two linearly independent forms of degree $2n + 2$ which vanish on $\mathbb{X}(n)$. Consequently,

$$\dim \sigma_{2n+1}\left(\mathbb{X}(n)\right) \leqslant N - 2$$

and hence it is defective.

In case $n = 1$, we showed in [CGG3, Example 2.2] that $\dim \sigma_3(\mathbb{X}(1)) = 13$. This is precisely $N - 2$ for this case. One can show that the ideal of $\sigma_3(\mathbb{X}(1))$ is generated by two quartics (hence these two) even though for $n = 1$ there is yet a third flattening which gives a third quartic in the ideal. But of these three quartics, any two generate the ideal and the third is a linear combination of the other two.

This raises several interesting questions for this family:

(1) Is $\sigma_{2n+1}(\mathbb{X}(n))$ the only defective secant variety for $\mathbb{X}(n)$? From [CGG2, Proposition 3.7] we know that $\sigma_s(\mathbb{X}(n))$ is not defective for $s \leqslant n + 1$.
(2) Is $\sigma_{2n+1}(\mathbb{X}(n))$ always the complete intersection of the two forms of degree $2n + 2$ that we found above?

Remark 2.11. Since the preprint which preceded this paper was distributed, [AOP] resolved the first question. They showed that the codimension of $\sigma_{2n+1}(\mathbb{X}(n))$ is exactly two. They also showed that it is the only defective secant variety in this family using their induction procedure. In fact, this last follows immediately from the knowledge that the codimension is exactly two.

Our reasoning, which differs from that in [AOP], goes as follows: given the codimension, one knows that the defectivity of the varieties $\sigma_{2n+1}(\mathbb{X}(n))$ is exactly 1 and hence that the secant varieties $\sigma_t(\mathbb{X}(n))$, $t \leqslant 2n$ (which must have smaller defectivity) cannot be defective at all. It is easy to check, using the fact that the codimension of $\sigma_{2n+1}(\mathbb{X}(n))$ is two, that $\sigma_{2n+2}(\mathbb{X}(n))$ is the entire enveloping projective space.

3. Segre–Veronese varieties

Up to this point we have only considered the Segre varieties, i.e. the embeddings of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ given by the very ample sheaves $\mathcal{O}(1, \ldots, 1)$. We can also consider the embeddings of these same varieties using the very ample sheaves $\mathcal{O}(d_1, \ldots, d_t)$, where $d_i > 0$.

These sheaves give a *Segre–Veronese* embedding (see [BM,CGG1]) into the projective space $\mathbb{P}^N$, where $N = (\prod_{i=1}^t N_i) - 1$ and where $N_i = \binom{n_i+d_i}{n_i}$. If we let $\underline{n} = (n_1, \ldots, n_t)$ and let $\underline{d} = (d_1, \ldots, d_t)$ then we will denote this embedding by $\phi_{\underline{n},\underline{d}}$ and its image by $\mathbb{X}_{\underline{n},\underline{d}}$. For other particular and recent results see [A,Ba,Bo].

If we denote by ν_{n_i,d_i} (or simply by ν_{d_i}, when no doubt can occur), the Veronese embedding of $\mathbb{P}^{n_i}$ using forms of degree d_i, then $\phi_{\underline{n},\underline{d}}$ is nothing more than the composition:

$$\psi \circ (\nu_{n_1,d_1}, \ldots, \nu_{n_t,d_t}) : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \to \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_t} \to \mathbb{P}^N$$

where ψ is just the usual Segre map and the other map is simply the product of the various Veronese embeddings.

To simplify the notation we just write:

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \xrightarrow{(d_1,\ldots,d_t)} \mathbb{P}^N.$$

In particular, we have that for $\underline{d} = (1, \ldots, 1)$, $\mathbb{X}_{\underline{n},(1,\ldots,1)}$ is a Segre variety, while, for $t = 1$, $\mathbb{X}_{\underline{n},\underline{d}}$ is the Veronese variety $\nu_{n,d}(\mathbb{P}^n)$.

Let M_i be the $(n_i + 1) \times \binom{n_i+d_i-1}{n_i}$ catalecticant matrix whose 2×2 minors define the ideal of the Veronese embedding $\nu_{d_i}(\mathbb{P}^{n_i})$ in $\mathbb{P}^{N_i}$. Consider the matrix $M = M_1 \otimes \cdots \otimes M_t$; since the $\nu_{d_i}(\mathbb{P}^{n_i})$'s are the rank 1 locus of M_i, $\mathbb{X}_{\underline{n},\underline{d}}$ is the rank 1 locus of M.

Since, for generic matrices, the locus of the $(s + 1) \times (s + 1)$ minors is precisely the variety $\sigma_s(\mathbb{Y})$, where $\mathbb{Y}$ is the locus of the 2×2 minors, it follows that $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ is contained in the zero locus of the $(s + 1) \times (s + 1)$ minors of M. We get, in this way, equations for $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$. In this case, however, the matrix M is not made up of independent coordinates (it has many equal entries, for example), and so we cannot "a priori" know the heights of the ideals given by its minors just knowing their size.

Nevertheless, whenever we expect $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ to fill $\mathbb{P}^N$, and yet the $(s + 1) \times (s + 1)$ minors of M give equations for $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$, we can definitely say that $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ is defective. To illustrate this consider the following two examples.

Example 3.1. Consider $t = 2$, $d_1 = d_2 = 2$ and $n_1 = n_2 = n$, i.e.

$$\mathbb{P}^n \times \mathbb{P}^n \xrightarrow{(2,2)} \mathbb{P}^N, \quad N = \binom{n+2}{2}^2 - 1.$$

Then for $n = 1, 2, 3$ we have that $\mathbb{X}_{\underline{n},\underline{d}}$ is *s-defective*, for $s = n^2 + 2n$.

In fact, for these cases M is an $(n + 1)^2 \times (n + 1)^2$ matrix, and thus its determinant is zero on $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$, for $s = n^2 + 2n$. Hence $\dim \sigma_{n^2+2n}(\mathbb{X}_{(n,n),(2,2)})$ is $\leqslant N - 1$. But, the expected dimension of $\sigma_s(\mathbb{X}_{(n,n),(2,2)})$ is $e = s(2n) + s - 1$, and a straightforward computation shows that $e \geqslant N$, for $n = 1, 2, 3$.

More precisely,

- $n=1$:

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{(2,2)} \mathbb{P}^8.$$

The image is the Del Pezzo surface $D_8 \subset \mathbb{P}^8$ and will be discussed in detail in the next section.

- $n=2$:

$$\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{(2,2)} \mathbb{P}^{35}.$$

Using, in a subtle manner, Horace's Method (see [CGG1]) we obtain: $\sigma_t(\mathbb{X}_{(2,2),(2,2)})$ has the expected dimension for $t \leqslant 6$, $t \geqslant 9$; $\sigma_8(\mathbb{X}_{(2,2),(2,2)})$ is a hypersurface whose equation is given above; the dimension of $\sigma_7(\mathbb{X}_{(2,2),(2,2)})$ is 32 rather than 34 (the expected dimension). The 8×8 minors of M give us equations in the ideal of $\sigma_7(\mathbb{X}_{(2,2),(2,2)})$ but we do not know if they generate that ideal.

- $n=3$:

$$\mathbb{P}^3 \times \mathbb{P}^3 \xrightarrow{(2,2)} \mathbb{P}^{99}.$$

Not only is $\mathbb{X}_{(3,3),(2,2)}$ 14-defective (using the determinant of M above) it is also 13-defective. We conjecture that all the other secant varieties of $\mathbb{X}_{(3,3),(2,2)}$ have the expected dimension.

Example 3.2. Consider $t=2$, $d_1=2k$, $d_2=2$, $n_1=1$ and $n_2=m$, i.e.

$$\mathbb{P}^1 \times \mathbb{P}^m \xrightarrow{(2k,2)} \mathbb{P}^N, \quad N=(2k+1)\binom{m+2}{2}-1.$$

Then $\forall m \geqslant 1, k \geqslant 1$, we have that $\mathbb{X}_{\underline{n},\underline{d}}$ is *s-defective*, for $s=km+k+m$. In fact, for these cases the $2k$-uple embedding of $\mathbb{P}^1$ is defined by a $(k+1)\times(k+1)$ matrix M_1, while the 2-uple embedding of $\mathbb{P}^m$ is defined by M_2 of size $(m+1)\times(m+1)$; hence $M=M_1\otimes M_2$ is an $(m+1)(k+1)\times(m+1)(k+1)$ matrix, and its determinant is zero on $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$, $s=mk+m+k$.

But (see [CGG1, §3]), for such a variety the value $s_0=km+k+\lceil\frac{m+1}{2}\rceil \leqslant km+k+m$, is the one for which we expect that $\sigma_{s_0}(\mathbb{X}_{\underline{n},\underline{d}})=\mathbb{P}^N$. Hence we have that $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ is s-defective for all s such that $s_0 \leqslant s \leqslant km+k+m$.

We would like to point out that all the examples we found in [CGG1, §3] can be viewed in this light, but this point of view also gives an equation for $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$. Indeed, for these examples we are able to find a single (determinantal) equation to demonstrate that $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ is defective. We conjecture that this equation is **the** defining equation for $\sigma_s(\mathbb{X}_{\underline{n},\underline{d}})$ and, as a consequence, that $\sigma_{s+1}(\mathbb{X}_{\underline{n},\underline{d}})=\mathbb{P}^N$.

If one considers "unbalanced" Segre–Veronese varieties, we have a result analogous to Theorem 2.4.

Theorem 3.3. *Let $\mathbb{X} = \mathbb{X}_{(n_1,\dots,n_t,n),(d_1,\dots,d_t,1)}$ be the Segre–Veronese embedding*

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \times \mathbb{P}^{n} \xrightarrow{(d_1,\dots,d_t,1)} \mathbb{X} \subset \mathbb{P}^M, \quad M = (n+1)\left(\prod_{i=1}^{t}\binom{n_i+d_i}{d_i}\right) - 1.$$

Let $\mathbb{X}'$ be the Segre–Veronese embedding

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \to \mathbb{P}^N, \quad N = \prod_{i=1}^{t}\binom{n_i+d_i}{d_i} - 1,$$

and let $\mathbb{Y}$ be the Segre embedding of $\mathbb{P}^N \times \mathbb{P}^n$ in $\mathbb{P}^M$. Let $n \geqslant N - \sum_{i=1}^t n_i + 1$.
Then:

(1) *for $2 \leqslant s \leqslant N - \sum_{i=1}^t n_i$, $\sigma_s(\mathbb{X}) \neq \sigma_s(\mathbb{Y})$ and $\sigma_s(\mathbb{X})$ has the expected dimension*;
(2) *for $s = N - \sum_{i=1}^t n_i + 1$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) \neq \mathbb{P}^M$ and $\sigma_s(\mathbb{X})$ has the expected dimension*;
(3) *for $N - \sum_{i=1}^t n_i + 1 < s \leqslant \min\{n, N\}$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) \neq \mathbb{P}^M$ and $\sigma_s(\mathbb{X})$ is defective with $\delta_{s-1}(\mathbb{X}) = s^2 - s(N - \sum_{i=1}^t n_i + 1)$*;
(4) *for $s \geqslant \min\{n, N\} + 1$, $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y}) = \mathbb{P}^M$*;
(5) *in cases* (2) *and* (3) *above, the ideal of $\sigma_s(\mathbb{X}) = \sigma_s(\mathbb{Y})$ is generated by the $(s+1) \times (s+1)$ minors of an $(n+1) \times (N+1)$ matrix of indeterminates.*

It follows that, in cases (2) *and* (3), *$\sigma_s(\mathbb{X})$ is a.C.M. and a minimal free resolution of its defining ideal is given by the Eagon–Northcott complex.*

Proof. Just as for Theorem 2.4, the proof here follows from Lemmas 2.2 and 2.3 (with $V = \mathbb{X}'$) and a calculation of the height of the ideal generated by the minors of a generic matrix of indeterminates. □

Corollary 3.4. *Let $\mathbb{X} = \mathbb{X}_{(n_1,\dots,n_t,n),(d'_1,\dots,d'_t,d)}$ be a Segre–Veronese variety such that*

$$d'_i \geqslant d_i \quad \textit{and} \quad n \geqslant \prod_{i=1}^{t}\binom{n_i+d_i}{d_i} - \sum_{i=1}^{t} n_i = N - \sum_{i=1}^{t} n_i + 1.$$

Then for every s such that $s \leqslant N - \sum_{i=1}^t n_i + 1$, we have that $\sigma_s(\mathbb{X})$ has the expected dimension.

Proof. By Theorem 3.3 we are done when $d = 1$. For $d > 1$, the result is an immediate consequence of [CGG1, Proposition 3.1].

Notice that, by analogy with Corollary 2.8, we can sometimes deduce Grassmann defectivity for Segre–Veronese varieties from Theorem 3.3. One would need some $d_i = 1$, $1 \leqslant i \leqslant t$.

Remark 3.5. Segre–Veronese varieties with two factors.

It is well known that $\mathbb{X}_{(m,n),(1,1)}$, the usual Segre embeddings of $\mathbb{P}^m \times \mathbb{P}^n$, has all its proper secant varieties defective. It is reasonable to wonder about the "next" natural family, $\mathbb{X}_{(m,n),(2,1)}$.

For $n \geqslant \binom{m+2}{2} - m = \binom{m+1}{2} + 1$ we know everything about the dimensions of the secant varieties from Theorem 3.3: the secant varieties for s small are not defective while those for s large are defective (until they fill the ambient space).

On the other hand, when $n = 1$ and m is arbitrary with $(d_1, d_2) = (2, 1)$ we have, [CGG1, Theorem 2.5], that there are no defective secant varieties.

Let us consider some particular examples for small m.

- $m = 1$: In this case we have, for any n, no defectivities among the secant varieties. Of course, these $\mathbb{X}_{(1,n),(2,1)}$ are all rational normal scrolls and so we also know the ideals of all their secant varieties.
- $m = 2$: When $m = 2$, $n \geqslant 4$, $(d_1, d_2) = (2, 1)$ we can apply Theorem 3.3 (as above). This leaves the three cases $\mathbb{X}_{(2,3),(2,1)}$, $\mathbb{X}_{(2,2),(2,1)}$ and $\mathbb{X}_{(2,1),(2,1)}$. Several calculations using the Methode d'Horace show that these varieties have no defective secant varieties.

4. A particular case: Del Pezzo surfaces

Here we want to investigate the ideals of the classically studied Del Pezzo surfaces and of their secant varieties. Let $S_9, S_8, \dots, S_3, D_8$ be the (smooth) Del Pezzo surfaces of degree d in $\mathbb{P}^d$, where $S_i \subset \mathbb{P}^i$ is obtained by blowing up $\mathbb{P}^2$ at $9 - i$ generic points, $i = 3, \dots, 9$ and then embedding this into $\mathbb{P}^i$ via the linear system given by the strict transforms of the cubic curves passing through the points. $D_8 \subset \mathbb{P}^8$, instead, is given by the embedding of a smooth quadric $Q \subset \mathbb{P}^3$ via the linear system given by $\mathcal{O}_Q(2)$, i.e. D_8 is the Segre–Veronese variety given by $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^8$ via $\mathcal{O}(2, 2)$.

Even though these varieties have been extensively studied in classical Algebraic Geometry, their ideals (or at least some of them) seem not to be widely known. For lack of a better reference we describe them here. All of them, except S_3, have ideals which are generated by quadrics (S_3 is a surface in $\mathbb{P}^3$ whose equation is the determinant of a 3×3 matrix of linear forms).

The ideal of S_4 is generated by two quadrics and the ideal of S_5 by three quadrics. To see how these equations can be obtained from the generators of the ideal of the points in $\mathbb{P}^2$ which have been blown up see e.g. [GiLo].

The ideal of S_9, which is the 3-uple (Veronese) embedding of $\mathbb{P}^2$ is well known; let $K[y_{000}, \dots, y_{ijk}, \dots y_{222}]$, $i, j, k \in \{0, 1, 2\}$, $i \leqslant j \leqslant k$ be the coordinate ring of $\mathbb{P}^9$ and let the embedding $\mathbb{P}^2 \to \mathbb{P}^9$ be the morphism associated to the map $K[y_{000}, \dots, y_{222}] \to R = K[x_0, x_1, x_2]$ such that $y_{ijk} \to x_i x_j x_k$. Then I_{S_9} is generated by the 2×2 minors of the catalecticant matrix, A_0, which describes the multiplication $R_1 \times R_2 \to R_3$:

$$A_0 = \begin{pmatrix} y_{000} & y_{001} & y_{002} & y_{011} & y_{012} & y_{022} \\ y_{001} & y_{011} & y_{012} & y_{111} & y_{112} & y_{122} \\ y_{002} & y_{012} & y_{022} & y_{112} & y_{122} & y_{222} \end{pmatrix}.$$

The ideals of S_8, S_7 and S_6 are known to be a.C.M. and generated by quadrics (e.g. see [Gi, GG]); we will check that these quadrics can be obtained as 2×2 minors of the matrix above just by erasing the last, then the fourth and then the first column of A_0.

In order to see why this is true we can view S_8 as given by the linear system of plane cubics containing $(0 : 0 : 1)$, i.e. those cubics whose defining equations does not contain the monomial x_2^3; hence S_8 is the projection of S_9 onto the $\mathbb{P}^8$ given by $y_{222} = 0$ from the point $(0 : \dots : 0 : 1) \in S_9$. Then we have that $I_{S_8} = I_{S_9} \cap K[y_{000}, \dots, y_{122}]$. Since we know that I_{S_8}

is generated in degree 2, its generators will be all the quadrics which are zero on S_9 and do not involve y_{222}. Those can be obtained by considering the 2×2 minors of the matrix A_1 obtained by erasing from A_0 the column containing y_{222}, and the trick is done! Actually, all minors of A involving the other two elements of that column are already given by the minors of A_1 and no linear combination of those involving y_{222} gives new quadrics in $K[y_{000}, \dots, y_{122}]$.

In the same way we get the matrices A_2 and A_3 (A_2 by erasing the fourth column from A_1 and then A_3 by erasing the first column from A_2) whose 2×2 minors give the ideals of S_7 and S_6. All this can also be easily checked by [CoCoA].

Notice that those determinantal ideals, except for that of S_6 (see also [GG]) are not generic, in the sense that they do not have the same height as the ideal of minors of a generic matrix of that size.

As for the ideal of D_8, working as in Example 3.1, we can see that it is generated by the 2×2 minors of the 4×4 matrix B (this too can also be easily checked via [CoCoA]):

$$B = \begin{pmatrix} y_{0000} & y_{0001} & y_{0100} & y_{0101} \\ y_{0001} & y_{0011} & y_{0101} & y_{0111} \\ y_{0100} & y_{0101} & y_{1100} & y_{1101} \\ y_{0101} & y_{0111} & y_{1101} & y_{1111} \end{pmatrix}.$$

Here the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^8$ is the one associated to the map

$$K[y_{0000}, \dots, y_{1111}] \to K[s_0, s_1 : t_0, t_1]; \quad y_{ijkl} \to s_i s_j t_k t_l, \quad \forall i, j, k, l \in \{0, 1\},\ i \leqslant j;\ k \leqslant l.$$

Now consider the varieties $\sigma_2(S_i)$, $i = 3, \dots, 9$. By Terracini's Lemma, we know that they have the expected dimension if the linear system of cubics passing through $9 - i$ points and two double points (all generic) have the expected dimension (i.e. $\max\{0, 9 - (9 - i) - 6\}$). It is well known (and easy to see) that this actually happens. So, all the S_i's are not 1-defective.

Also for S_3, S_4 and S_5 we have $\sigma_2(S_i) = \mathbb{P}^i$. So, there is nothing to say about the ideal of $\sigma_2(S_i)$, $i = 3, 4, 5$.

When $i = 6, 7, 8, 9$ we want to show that the ideal of $\sigma_2(S_i)$ is generated by the 3×3 minors of the matrices A_{9-i} above.

First observe that, by a result of Kanev (see [K]), the ideal of $\sigma_2(S_9)$ is given by the 3×3 minors of A_0. Now consider the following remark:

Remark 4.1. "The secant variety of a projection is the projection of the secant variety." Let $X \subset \mathbb{P}^n$ be a non-degenerate reduced and irreducible variety, and $P \in \mathbb{P}^n$; let $\pi_P : \mathbb{P}^n - \{P\} \to H$ be the projection from P to a generic hyperplane $H \cong \mathbb{P}^{n-1}$ and $X' = \overline{\pi_P(X - P)}$. Then $\sigma_2(X') = \overline{\pi_P(\sigma_2(X) - P)}$.

In fact, the inclusion $\sigma_2(X') \subset \overline{\pi_P(\sigma_2(X) - P)}$ is obvious. As for the other inclusion, let Q be a generic point of $\pi_P(\sigma_2(X) - P)$, and $Q' \in \sigma_2(X) - P$ a point in its preimage: there will be a secant line L to X, not containing P by genericity, which contains Q', hence $\pi_P(L)$ will be a secant line of $\pi_P(X - P)$, and $Q \in \sigma_2(X')$.

Now, since S_8, S_7 and S_6 are obtained, each from the previous, by projection from one point at a time (starting from S_9), the same is true for their secant varieties (Remark 4.1). Using elimination, as we did for the ideals of S_8, S_7, S_6 themselves, we see that the ideals of $\sigma_2(S_i)$, $i = 6, 7, 8$ are given by the 3×3 minors of the matrices A_{9-i} (again, one can check this using [CoCoA]).

Notice that in this case, i.e. for $\sigma_2(S_8), \sigma_2(S_7), \sigma_2(S_6)$ those determinantal ideals have generic height, hence are arithmetically Cohen–Macaulay with known resolution given by the Eagon–Northcott complex.

For $\sigma_2(D_8)$ we have to consider the $(2,2)$ divisors through two generic 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. It is easy to check that this linear system has the expected dimension $(= 3)$, hence $\sigma_2(D_8)$ has the expected dimension $(= 5)$.

Is the ideal of $\sigma_2(D_8)$ generated by the 3×3 minors of B? By using [CoCoA] one can check that this is the case (notice that it does not have generic height).

From [CS] we know the degree of $\sigma_2(D_8)$: $\deg \sigma_2(D_8) = 10$. Notice also that the degree of $\sigma_2(S_8)$ is 10. A quick and easy check with **CoCoA** shows that all the graded Betti numbers of these two varieties are also equal. Hence up to this point S_8 and D_8 (and also their chordal varieties) cannot be distinguished by numerical invariants.

As for $\sigma_2(S_9)$, its degree is 15 (by [CS] again) and this agrees with the fact that its ideal is generated by the 3×3 minors of A_0 (this was known by [K]).

From Example 3.1 we know that $\sigma_3(D_8)$ does not have the expected dimension. It should fill $\mathbb{P}^8$, but instead it is a hypersurface (see [CGG1]). Moreover, the equation of $\sigma_3(D_8)$ is given by $\det B$.

This is an interesting difference between D_8 and S_8: $\sigma_3(S_8)$ fills up $\mathbb{P}^8$ as expected, since there are no cubics in $\mathbb{P}^2$ passing through three double points and a simple one.

In the same way we get that $\sigma_3(S_i) = \mathbb{P}^i$ for $i = 6, 7$, instead $\sigma_3(S_9)$ is a hypersurface, as expected. Actually, $\sigma_3(S_9)$ is the hypersurface parameterizing Fermat cubics, so its equation (of degree 4) is defined by the Aronhold (or Clebsh) invariant of a cubic (e.g. see [Ge] or [DK]).

Acknowledgments

After this paper was written T. Abo, G. Ottaviani and C. Peterson obtained results about the *dimensions* of the secant varieties of products of projective spaces which are the same as our results on the dimensions in Theorem 2.4 (see [AOP]). We are very grateful to them for bringing their work to our attention. We also thank E. Carlini for his help in making many of the computer calculations on which our conjectures and results are based.

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